A Combinatorial Characterization of the Critical Itineraries of an Overlapping Dynamical System

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Abstract

A uniform overlapping function of the unit interval to itself is a piecewise continuous function with a single point of discontinuity and with two linear branches of the same slope. The itineraries of the point of discontinuity, which arise in the study of fractal transformations, are called the critical itineraries. A combinatorial characterization of the critical itineraries of uniform overlapping functions is provided.

1 Introduction

The dynamics of a function from the unit interval to itself is a topic with a long and fascinating history. Although most results concern continuous functions, there is research, especially in electronics applications, on the dynamics of discontinuous functions [5, 6, 7, 9, 10]. Recently, the dynamics of discontinuous functions have played a role in the study of fractal transformations [2, 3]. This paper concerns the dynamics of a piecewise continuous function of the simple form shown in Figure 1. More specifically,

$$f_{(a,p,-)}(x) = \begin{cases} ax & \text{if} \quad x \leq p \\ ax + (1-a) & \text{if} \quad x > p, \end{cases}$$

or

$$f_{(a,p,+)}(x) = \begin{cases} ax & \text{if} \quad x < p \\ ax + (1-a) & \text{if} \quad x \geq p, \end{cases}$$

where $1 < a \leq 2$ and $1 - \frac{1}{a} \leq p \leq \frac{1}{a}$. Call a function of this form **uniform overlapping**. It is uniform in the sense that the two branches are linear with the same slope, and it is overlapping in the sense that $f([0,p] \cap f([p,1] \neq \emptyset$. Computing the entropy of overlapping functions (not necessarily uniform) is the subject of the papers [3, 11]. The notation $\pm$

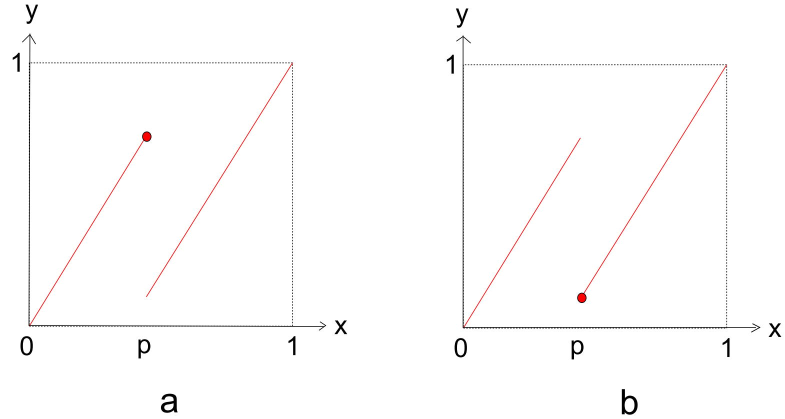

Figure 1: A uniform overlapping dynamical system.

will mean either $-$ or $+$, and the subscript $(a, p, \pm)$ is omitted when no confusion arises. Also we will omit the term 'uniform' since, in this paper, all overlapping functions are uniform. Let $f^n = f \circ f \circ \cdots \circ f$ denote the n^{th} iterate of f. The following "primality tester" is a special case the main theorem of this paper.

- There is an $a \approx 1.79$ and a $p \approx .44$ with the following property: the natural number n is prime if and only if $f^{(n-1)}_{(a,p,-)}(p) > p$.

In other words, to test whether n is prime, we iterate the point of discontinuity $n-1$ times. If this itererate lies to the right of p, then n is prime; if it lies to the left, then composite. Two comments are in order. First, this result has little to do with number theory. Second, the method is numerically problematic because a and p, being irrational numbers, can be estimated to at most finitely many places.

The main results in this paper, of which the primality tester is a special case, are as follows. For an overlapping function, let I_0 denote the left interval $[0, p]$ in the case of Figure 1a and $[0, p)$ in the cae of Figure 1b. Let I_1 denote the right interval $(p, 1]$ in the case of Figure 1a and $[p, 1]$ in the cae of Figure 1b. Let $\Omega = \{0, 1\}^\infty$ denote the set of infinite strings $\omega_0\omega_1\omega_1 \cdots$ using symbols $0, 1$. The two **itinerary maps** $\tau_{(a,p,\pm)} : [0, 1] \to \Omega$ of $f := f_{(a,p,\pm)}$ are given by $\tau_{(a,p,\pm)}(x) = \omega := \omega_0\,\omega_1\,\omega_2 \cdots \in \Omega$, where

$$\omega_n = \begin{cases} 0 & \text{if} \quad f^n(x) \in I_0 \\ 1 & \text{if} \quad f^n(x) \in I_1. \end{cases}$$

For $x \in [0, 1]$, the image $\tau_{(a,p,\pm)}(x)$ is called the **itinerary** of x. The two itineraries (- and $+$) of the point of discontinuity p will play a special role in this paper. Call $\tau_- := \tau_{(a,p,-)}(p)$ and $\tau_+ := \tau_{(a,p,+)}(p)$ the **critical itineraries** of $f_{(a,p,-)}$ and $f_{(a,p,+)}$, respectively. The first part of the following theorem is a combinatorial characterization of the critical itineraries, the combinatorial conditions *admissible* and *null* defined in Section 2.

Theorem 1. *Strings α and β in Ω are the critical itineraries of some overlapping function if and only if the pair (α, β) is admissible and non-null. Moreover, if (α, β) is admissible and non-null, then α and β are the critical itineraries of the overlapping*

functions $f_{(1/r,p,-)}$ *and* $f_{(1/r,p,+)}$*, respectively, where* r *is the smallest solution to the equation*

$$\sum_{n=0}^{\infty} \alpha_n\, x^n = \sum_{n=0}^{\infty} \beta_n\, x^n$$

in the interval $[0,1)$ *and* $p = \sum_{n=0}^{\infty} \alpha_n\, r^n$.

Example 1. The pair (α, β):

$$\begin{aligned} \alpha &= 0\,1\,1\,0\,1\,0\,1\,0\,0\,0\,1\,0\,1 \cdots \\ \beta &= 1\,0\,0\,0 \cdots\,, \end{aligned}$$

where $\alpha_n = 1$ if and only if $n+1$ is prime, is admissible and non-null (see Example 3). Therefore Theorem 1 asserts that α and β are the critical itineraries of an overlapping function f. By the definition of the critical itineraries, it is exactly this function that provides the previously described primality tester. A calculation using Theorem 1 implies that the overlapping function is $f_{(a,p,-)}$, where a is approximately 1.792568768 and p is approximately .4421413462.

The remainder of this paper contains the proof of Theorem 1. The definitions of the terms admissible and null appear in Section 2. Also in that section is the proof that (α, β) addmissible and non-null are necessary conditions for α and β to be the critical itineraries of some overlapping function. The proof that they are also sufficient conditions appears in Sections 3 and 4. Section 3 introduces the notion of the projection map and contains results on the projection map required for the proof of Theorem 1. Section 4 concludes the proof.

2 Admissible and Null Pairs

For a string $\omega = \omega_0\, \omega_1\, \omega_2 \cdots \in \Omega$, the $(n+1)^{st}$ element in the string is denoted ω_n, and the initial string $\omega_0\, \omega_1\, \omega_2 \cdots \omega_n$ is denoted $\omega|_n$. For $\Gamma \subseteq \Omega$, let $\Gamma_n = \{\omega|_n \,:\, \omega \in \Gamma\}$. A line over a finite string denotes infinite repetition, for example $\overline{01} = 010101 \cdots$.

Let S denote the shift operator on Ω, i.e, $S(\omega_0\, \omega_1\, \omega_2 \cdots) = \omega_1\, \omega_2, \omega_3 \cdots$. Also S^n denotes the n^{th} iterate of S.

The lexicographic order $\preceq$ on Ω is the total order defined by $\sigma \prec \omega$ if $\sigma \neq \omega$ and $\sigma_k < \omega_k$ where k is the least index such that $\sigma_k \neq \omega_k$. We use the notation

$$[\alpha, \beta] = \{\omega \in \Omega \,:\, \alpha \preceq \omega \preceq \beta\}$$

for the closed interval; likewise for the half open intervals $[\alpha, \beta)$ and $(\alpha, \beta]$.

Definition 1. Call a pair (α, β) of strings in Ω **admissible** if it satisfies

1. $\alpha_0 = 0,\ \alpha_1 = 1 \quad$ and $\quad \beta_0 = 1,\ \beta_1 = 0$, and
2. $S^n\alpha \notin (\alpha, \beta] \quad$ and $\quad S^n\beta \notin [\alpha, \beta) \quad$ for all $n \geq 0$.

The **address spaces associated with an admissible pair** (α, β) are

$$\begin{aligned}\Omega_{(\alpha,\beta,-)} &:= \{\omega \in \Omega \,:\, S^n\omega \notin (\alpha,\beta] \text{ for all } n \geq 0\} \\ \Omega_{(\alpha,\beta,+)} &:= \{\omega \in \Omega \,:\, S^n\omega \notin [\alpha,\beta) \text{ for all } n \geq 0\} \\ \Omega_{(\alpha,\beta)} &:= \Omega_{(\alpha,\beta,-)} \cup \Omega_{(\alpha,\beta,+)}.\end{aligned}$$

There exist admissible pairs (α, β) whose address spaces are exceptionally sparse in the following sense. For $\Gamma \subseteq \Omega$, let $|\Gamma|$ denotes the cardinality of Γ. Define the **exponential growth rate** $h(\Gamma)$ of $\Gamma \subseteq \Omega$ by

$$h(\Gamma) = \limsup_{n\to\infty} \frac{1}{n} \ln|\Gamma_n|.$$

Although the address spaces usually have positive exponential growth rate, it is possible that $h(\Omega_{(\alpha,\beta)}) = 0$, as demonstrated by the following example.

Example 2. The following α and β consititute an admissible pair:

$$\begin{aligned}\alpha &= 01\overline{10} \\ \beta &= 10\overline{01}.\end{aligned}$$

First notice that a string $\omega \in \Omega_{(\alpha,\beta)}$ cannot cantain either 1000 or 0111 as a substring. Moreover ω cannot contain either $100\,s\,00$ or $011\,t\,11$ as a substring, where $s = 10101\cdots1$ and $t = 01010\cdots0$. In other words, ω is a string of alternating $0's$ and $1's$ with possibly one double zero or one double one, and possibly a string of zeros or a string of ones at the beginning. With these rigid restrictions, it is not hard to check that $h(\Omega_{(\alpha,\beta)}) = 0$.

Definition 2. If $h(\Omega_{(\alpha,\beta)}) = 0$, call the pair (α, β) **null**.

Example 3. The pair of Example 1 is admissible and not null. It is easy to check that the pair is admissible. Concerning non-null, in this case the set $\Omega_{(\alpha,\beta)}|_n$ contains all strings of length $n+1$ of the form $(s_1)\,1\,(s_2)\,1\,(s_3)\,1\cdots$, where (s_k) is any finite string of (at least one) zeros. So it is not hard to check that $h(\Omega_{(\alpha,\beta)}) > 0$.

For an admissible pair (α, β) to be null is not typical. For example, if (α, β) is an admissible pair for which $\alpha = 011\alpha'$ and $\beta = 100\beta'$ and either α' begins wilth a 1 or β' begins with a 0, then (α, β) is non-null. This is because, in the case that α' begins wilth a 1 say, any string of the form $0(s_1)\,0\,(s_2)\,0\,(s_3)\,0\cdots \in \Omega_{(\alpha,\beta)}$, where $(s_n) = 1$ or $(s_n) = 11$ for $n = 1, 2, \ldots$. It may be an interesting question to characterize the set of admissible pairs that are null.

Define the two **address spaces** of the pair $f_{(a,p,\pm)}$ of overlapping functions by

$$\Omega_{(a,p,\pm)} := \tau_{(a,p,\pm)}([0,1]).$$

Theorem 2. *If $\alpha, \beta \in \Omega$ are the critical itineraries of an overlapping function, then (α, β) is a non-null, admissible pair.*

Proof. In somewhat greater generality the following statement is proved in [3, Theorem 5.2]. For an overlapping function with critical itineraries τ_- and τ_+, the address spaces are given by

$$\Omega_{(a,p,+)} = \{\omega \in \Omega \,:\, S^n\omega \notin [\tau_-,\tau_+) \quad \text{for all} \quad n \geq 0\}$$
$$\Omega_{(a,p,-)} = \{\omega \in \Omega \,:\, S^n\omega \notin (\tau_-,\tau_+] \quad \text{for all} \quad n \geq 0\}.$$

Since $\beta \in \Omega_{(a,p,+)}$ and $\alpha \in \Omega_{(a,p,-)}$, it is immediate that $S^n\beta \notin [\alpha,\beta)$ for all $n \geq 0$ and $S^n\alpha \notin (\alpha,\beta]$ for all $n \geq 0$. Therefore the pair is admissible.

The function $f_{(a,p,\pm)}$ acting on the unit interval may be considered as a dynamical system. For such a dynamical system there is the notion of the topological entropy, a concept introduced by Adler, Konheim and McAndrews [1] as a measure of the complexity of the dynamical system. Bowen [4] gave an equivalent definition of entropy for a continuous map of a metric space. For our purposes the following definition of the entropy $h(f)$ of $f := f_{(a,p,\pm)}$, based on that given by Bowen, suffices:

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log |[\Omega_{(a,p,-)}]_n| = \lim_{n\to\infty} \frac{1}{n} \log |[\Omega_{(a,p,+)}]_n|.$$

It is known [8] that the topological entropy of $f_{(a,p,\pm)}$ is $\ln(a) > 0$. If, however, $\alpha, \beta \in \Omega$ are the critical itineraries of some overlapping functions $f_{(a,p,\pm)}$, then $\Omega_{(\alpha,\beta,\pm)} = \Omega_{(a,p,\pm)}$, and therefore $h(\Omega_{(\alpha,\beta)}) = h(\Omega_{(a,p,\pm)}) > 0$. □

3 Projection Map

Definition 3. For $x \in [0,1)$ and $\omega \in \Omega$, the **projection map** $\pi_x : \Omega \to [0,1]$ is given by

$$\pi_x(\omega) = (1-x)\sum_{k=0}^{\infty} \omega_k\, x^k.$$

Proposition 1 below contains a couple of known [3] properties of the projection map, one of which concerns continuity. The relevant metric on the space of strings is as follows. The set Ω of binary strings is a compact metric space with respect to the metric given by

$$d(\omega,\sigma) = \begin{cases} 2^{-k} & \text{if } \omega \neq \sigma \\ 0 & \text{if } \omega = \sigma, \end{cases}$$

where k is the least index such that $\omega_k \neq \sigma_k$.

Proposition 1. *If $a \in (1,2]$, then*

1. *$\pi_x(\omega)$ is a continuous function of $\omega \in \Omega$ with x fixed and a continuous function of $x \in [0,1)$ with ω fixed, and*
2. *if $g_0(x) = \frac{1}{a}\,x$ and $g_1(x) = \frac{1}{a}\,x + (1-\frac{1}{a})$, then*

$$\pi_a(\omega) = \lim_{n\to\infty} (g_{\omega_0} \circ g_{\omega_1} \circ \cdots \circ g_{\omega_n})(x_0).$$

for all $\omega \in \Omega$, the limit independent of the value of x_0. (Note that g_0 and g_1 are the inverse functions of the two branches of the overlapping functions $f_{(a,p,\pm)}$.)

Throughout this section we assume that (α, β) is admissible and non-null. The main result in this section, Theorem 3, states that $\pi_x(\alpha) = \pi_x(\beta)$ for some $x \in (0,1)$.

Lemma 1. *If there is no $x \in (0,1)$ such that $\pi_x(\alpha) = \pi_x(\beta)$, then $\pi_x : \Omega_{(\alpha,\beta)} \to [0,1]$ is strictly increasing for all $x \in (0,1)$.*

Proof. Assume, by way of contradiction, that there is an $x \in (0,1)$ such that $\pi_x : \Omega_{(\alpha,\beta)} \to [0,1]$ is not strictly increasing. Then there exists $\sigma, \omega \in \Omega_{(\alpha,\beta)}$ such that $\sigma \prec \omega$ and $\pi_x(\sigma) \geq \pi_x(\omega)$. By removing the maximum initial portion of the strings where σ and ω are equal, it may also be assumed, without loss of generality, that $\sigma_0 = 0$ and $\omega_0 = 1$. Then, since $\pi_0(\sigma) = 0 < 1 = \pi_0(\omega)$, the continuity of π_x stated in Proposition 1, insures that there is an $x \in (0,1)$ such that $\pi_x(\sigma) = \pi_x(\omega)$. Let

$$r = \min\{x \in (0,1) \,:\, \pi_x(\sigma) = \pi_x(\omega) \text{ for some } \sigma, \omega \in, \Omega_{(\alpha,\beta)},\, \sigma_0 = 0,\, \omega_0 = 1\}. \tag{1}$$

Note that $\pi_{1/3} : \Omega \to [0,1]$ is strictly increasing. This is easily seen because $\pi_{1/3}(\omega)$ is (up to the constant factor $2/3$) just the real number represented by the base 3 decimal $.\omega_0\omega_1\omega_2\cdots$. (In base 3 the decimals $.0222\cdots$ and $.1000\cdots$ are equal, but that does not concern us because ω has only terms 0 and 1.) In particular, because $\sigma, \omega \in \Omega_{(\alpha,\beta)}$ implies $\sigma \preceq \alpha \prec \beta \preceq \omega$, we have

$$\pi_{\frac{1}{3}}(\sigma) \leq \pi_{\frac{1}{3}}(\alpha) < \pi_{\frac{1}{3}}(\beta) \leq \pi_{\frac{1}{3}}(\omega). \tag{2}$$

Now consider $\pi_x(\sigma), \pi_x(\alpha), \pi_x(\beta), \pi_x(\omega)$ as functions of $x \in [1/3, a]$. (It is helpful to visualize the graphs of these these four functions.) The continuity of π_x with respect to x and the assumption that there is no $x \in (0,1)$ such that $\pi_x(\alpha) = \pi_x(\beta)$, forces the existence of a number $b < r$ such that either $\pi_b(\alpha) = \pi_b(\sigma)$ or $\pi_b(\beta) = \pi_b(\omega)$. (If it happens that $\alpha = \sigma$, then $\pi_b(\beta) = \pi_b(\omega)$, and if it happens that $\beta = \omega$, then $\pi_b(\alpha) = \pi_b(\sigma)$. Both $\alpha = \sigma$ and $\beta = \omega$ is not possible.) In either case, after removing the maximum initial portion of the strings where they are equal, we arrive at a contradiction to the minimality of r. $\square$

Lemma 2. *Assume that there is no $x \in (0,1)$ such that $\pi_x(\alpha) = \pi_x(\beta)$. For every $a \in (1,2]$, there is a p such that $1 - 1/a \leq p \leq 1/a$ and such that the overlapping functions $f_{(a,p,\pm)}$ satisfies $f_{(a,p,\pm)}(\pi_{1/a}\omega) = \pi_{1/a}(S\omega)$ for all $\omega \in \Omega_{(\alpha,\beta,+)}$.*

Proof. We give the proof for $+$; the proof for $-$ is similar. For ease of notation let $f := f_{(a,p,+)}$ and let $f_0(x) = ax$ and $f_1(x) = ax + (1-a)$. Let $x := 1/a$ and let p be any real number such $\pi_x(\alpha) \leq p \leq \pi_x(\beta)$ and $1 - 1/a \leq p \leq 1/a$. To show that there exists such a p with $1 - 1/a \leq p \leq 1/a$, it is sufficient to show that $\pi_x(\alpha) \leq 1/a$ and $\pi_x(\beta) \geq 1 - 1/a$. This is done as follows:

$$\begin{aligned}\frac{1}{a}\,\pi_x(\alpha) = f_0\left(\pi_x(\alpha)\right) &= f_0((1-x)\sum_{n=0}^{\infty}\alpha_n x^n) = (1-x)\sum_{n=0}^{\infty}\alpha_{n+1}x^n + \frac{\alpha_0}{x} \\ &= (1-x)\sum_{n=0}^{\infty}\alpha_{n+1}x^n = \pi_x(S\alpha) \leq 1,\end{aligned}$$

$$\frac{1}{a}\,\pi_x(\beta) + (1-\frac{1}{a}) = f_1(\pi_x(\beta)) = f_1((1-x)\sum_{n=0}^{\infty}\beta x^n) = (1-x)\sum_{n=0}^{\infty}\beta_{n+1}x^n + \frac{\beta_0}{x} - \frac{1}{x}$$
$$= (1-x)\sum_{n=0}^{\infty}\beta_{n+1}x^n = \pi_x(S\beta) \geq 0.$$

To finish the proof, assume that $\omega \in \Omega_{(\alpha,\beta,+)}$. From Lemma 1, if $\pi_x(\omega) < p \leq \pi_x(\beta)$, then $\omega \prec \beta$. Since $\omega \notin [\alpha,\beta)$, it follows that $\omega \prec \alpha$, and hence $\omega_0 = 0$. Likewise if $\pi_x(\omega) \geq p \geq \pi_x(\alpha)$, then $\omega \succeq \alpha$. Since $\omega \notin [\alpha,\beta)$, it follows that $\omega \succeq \beta$ and hence $\omega_0 = 1$. Therefore if $\pi_x(\omega) < p$ then

$$f\,(\pi_x(\omega)) = f_0((1-x)\sum_{n=0}^{\infty}\omega_n x^n) = (1-x)\sum_{n=0}^{\infty}\omega_{n+1}x^n + \frac{\omega_0}{x}$$
$$= (1-x)\sum_{n=0}^{\infty}\omega_{n+1}x^n = \pi_x(S\omega),$$

and if $\pi_x(\omega) \geq p$, then

$$f(\pi_x(\omega)) = f_1((1-x)\sum_{n=0}^{\infty}\omega x^n) = (1-x)\sum_{n=0}^{\infty}\omega_{n+1}x^n + \frac{\omega_0}{x} - \frac{1}{x}$$
$$= (1-x)\sum_{n=0}^{\infty}\omega_{n+1}x^n = \pi_x(S\omega).$$

□

Lemma 3. *Assume that there is no $x \in (0,1)$ such that $\pi_x(\alpha) = \pi_x(\beta)$. For any $a \in (1,2]$, let p be the number provided in Lemma 2. If $f := f_{(a,p,+)}$ is the corresponding overlapping function, then $\Omega_{(\alpha,\beta,+)} \subseteq \Omega_{(a,p,+)}$.*

Proof. Given $\omega \in \Omega_{(\alpha,\beta,+)}$, let $t = 1/a$ and $y = \pi_t(\omega)$. Further let $f_0(x) = ax$ and $f_1(x) = ax + (1-a)$, and recall that f_0, f_1 are the inverses of the functions g_0, g_1 in statement 4 of Proposition 1. Then by Lemma 2 and statement 4 of Proposition 1

$$\begin{aligned} f^n(y) &= \pi_t(S^n\omega) = \pi_t(\omega_n\,\omega_{n+1}\cdots) = (f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1} \circ f_{\omega_0})(\pi_t(\omega)) \\ &= (f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1} \circ f_{\omega_0})(y) \end{aligned}$$

for all $n \geq 0$. By the definition of itinerary, this implies that $\tau_{(a,p,+)}(y) = \omega$, and hence $\omega \in \Omega_{(a,p,+)}$. □

Theorem 3. *If (α,β) is admissible and non-null, then there exists an $x \in (0,1)$ such that $\pi_x(\alpha) = \pi_x(\beta)$.*

Proof. Assume, by way of contradiction, that there is no such $x \in (0,1)$. According to Lemma 3, for every $a \in (1,2]$, there is an overlapping function $f_{(a,p,+)}$ such that $\Omega_{(\alpha,\beta,+)} \subseteq \Omega_{(a,p,+)}$. However, from the definitions of the expondential growth rate of and topological entropy (in the proof of Theorem 2) it follows that $h(\Omega_{(\alpha,\beta)} \leq h(f_{(a,p,+)}) = \ln(a)$. Letting $a \to 1$, this implies that $h(\Omega_{(\alpha,\beta)} = 0$, contradicting that the pair (α,β) is non-null. □

4 Main Theorem

Theorem 1 will follow immediately from Corollary 2 and Theorem 4 below. In light of Theorem 3 define

$$r := \min\{x \in (0,1) \,:\, \pi_x(\alpha) = \pi_x(\beta).$$

Lemma 4. *Assume (α, β) is admissible and non-null. The map $\pi_r \,:\, \Omega_{(\alpha,\beta,-)} \cup \Omega_{(\alpha,\beta,+)} \to [0,1]$ is increasing, and the maps $\pi_r \,:\, \Omega_{(\alpha,\beta,-)} \to [0,1]$ and $\pi_r \,:\, \Omega_{(\alpha,\beta,+)} \to [0,1]$ are strictly increasing.*

Proof. The proof that $\pi_x \,:\, \Omega_{(\alpha,\beta,-)} \cup \Omega_{(\alpha,\beta,+)} \to [0,1]$ is strictly increasing for all $x < r$ is exactly the same as the proof of Lemma 1. The first statement of this lemma then follows from the continuity of $\pi_x(\omega)$ is both x and ω (Proposition 1).

Concerning the second statement of Lemma 4, we start with some preliminary observations. Let Q_0, Q_1 denote the set of two intervals obtained from Ω by removing the open interval (α, β), and let $Q^1 := Q_0 \cup Q_1$. The first element of any sting in Q_0 is 0, and the first element in any sting in Q_1 is 1. Likewise, let Q_{00} and Q_{01} be obtained from Q_0 by removing the interval $(0\alpha, 0\beta)$, and let Q_{01} and Q_{11} be obtained from Q_1 by removing the interval $(1\alpha, 1\beta)$. Define $Q^2 := Q_{00} \cup Q_{01} \cup Q_{10} \cup Q_{11}$. Continue in this way to obtain Q^n, $n \geq 1$. To be more precise, we illustrate using $Q_{01} = [0\beta, \alpha]$ as an example. There is a greatest integer k such that $s := (0\beta)|_k = \alpha|_k$. Now Q_{s0} and Q_{s1} are obtained from Q_{01} by removing the interval $(s\alpha, s\beta)$. (Note that there may be many finite strings t for which Q_t does not exists.) In general, Q^n is the union of all the Q_t, where t is a string of length t. The sequence $\{Q^n,\, n \geq 1\}$ of sets is nested and $\bigcap_{n=1}^{\infty} Q^n = \Omega_{(\alpha,\beta)}$. The left and right endpoints of each interval Q_s lie in $\bigcap_{n=1}^{\infty} Q^n$ and hence in $\Omega_{(\alpha,\beta)}$. Moreover, if the last two elements in the sting s are 01, then $Q_s = [t\,0\beta, t\,\alpha]$, and if the last two elements in the sting s are 10, then $Q_s = [t\,\beta, t\,1\alpha]$ for some string t of length $n-2$. Concerning the endpoints of these intervals, the fact that $\pi_a(\alpha) = \pi_a(\beta)$ implies that

$$\begin{aligned} &\pi_r(t\alpha) - \pi_r(t\,0\beta) = r^{n-2}(\pi_r(\alpha) - r\pi_r(\beta)) = r^{n-2}(1-r)\pi_r(\alpha) > 0, \quad \text{and} \\ &\pi_r(t\,1\alpha) - \pi_r(t\beta) = r^{n-2}((1-r) + r\pi_r(\alpha) - \pi_r(\beta)) = r^{n-2}(1-r)(1-\pi_r(\alpha)) > 0. \end{aligned} \tag{1}$$

Returning to the proof of the lemma, assume $\sigma, \omega \in \Omega_{(\alpha,\beta)}$ and $\sigma \prec \omega$. Since the length (in the metric on Ω) of the intervals Q_s tends to 0 as the length of s tends to infinity, there is an interval of the form $Q_s = [\mu, \nu]$ properly between σ and ω in the lexicographic order, where $[\mu, \nu] = [t\,0\beta, t\,\alpha]$ or $[\mu, \nu] = [t\,\beta, t\,1\alpha]$. Since $t\,0\beta, t\,\alpha, t\,\beta, t\,1\alpha \in \Omega_{(\alpha,\beta)}$, the first statement in Lemma 4 and Equation 1 now imply

$$\pi_a(\sigma) \leq \pi_a(\mu) < \pi_a(\nu) \leq \pi_a(\omega).$$

□

Theorem 4. *Let (α, β) be admissible and non-null. If $r = \min\{x \in (0,1) \,:\, \pi_x(\alpha) = \pi_x(\beta)\}$ and $p = \pi_r(\alpha) = \pi_r(\beta)$, then α and β are the critical itineraries of the overlapping functions $f_{(1/r,p,\pm)}$.*

Proof. Let $a = 1/r$. To insure that $f_{(a,p,\pm)}$ are overlapping functions, we must first show that $1 - r \leq p \leq r$. But

$$1 - r \leq (1-r)\sum_0^{\infty} \beta_n r^n = p = (1-r)\sum_0^{\infty} \alpha_n r^n \leq (1-r)\sum_1^{\infty} r^n = r.$$

We will prove that $\tau_+ := \tau_{(a,p,+)}(p) = \beta$; the proof that $\tau_- := \tau_{(a,p,-)}(p) = \alpha$ is essentially the same. Let $f := f_{(a,p,+)}$ and let $\omega := \tau_{(a,p,+)}(p)$. For all $n \geq 0$, we will prove the following by induction on n:

(i) $\omega_n = \beta_n$ and,

(ii) $f^n(p) = \pi_r(S^n\beta)$.

Since both β and ω begin with a 1, both statements are true for $n = 0$. Assuming the two statements true for $n-1$, we will prove that they are true for n.

Starting with statement (ii):

$$\begin{aligned} f^n(p) &= f(f^{n-1}(p)) = f(\pi_r(S^{n-1}\beta)) = f((1-r)\sum_{k=0}^{\infty}\beta_{n-1+k}r^k) \\ &= (1-r)\sum_{k=0}^{\infty}\beta_{n+k}r^k = \pi_r(S^n\beta). \end{aligned}$$

The second to last equality above comes from the following direct calculation: if $\omega_{n-1} = 0$, then by the induction hypothesis $\beta_{n-1} = 0$ and

$$f((1-r)\sum_{k=0}^{\infty}\beta_{n-1+k}r^k) = (1-r)\sum_{k=0}^{\infty}\beta_{n+k}r^k + \frac{\beta_{n-1}}{r} = (1-r)\sum_{k=0}^{\infty}\beta_{n+k}r^k = \pi_r(S^n\beta),$$

and, if $\omega_{n-1} = 1$, then $\beta_{n-1} = 1$ and

$$f((1-r)\sum_{k=0}^{\infty}\beta_{n-1+k}r^k) = (1-r)\sum_{k=0}^{\infty}\beta_{n+k}r^k + \frac{\beta_{n-1}}{a} - \frac{1}{a} = (1-r)\sum_{k=0}^{\infty}\beta_{n+k}r^k = \pi_r(S^n\beta).$$

Concerning statement (i), if $\beta_n = 0$ then, by assumption 2 in the statement of this theorem, we have $S^n\beta \prec \alpha$. Because $S^n\beta \in \Omega_{(\alpha,\beta,-)}$ and $\alpha \in \Omega_{(\alpha,\beta,-)}$, Lemma 4 applies. By that lemma and by statement (ii), which we have just proved, we have $f^n(p) = \pi_r(S^n\beta) < \pi_r(\alpha) = p$. By the definition of the itinerary of p this implies that $\omega_n = 0$, and hence $\omega_n = \beta_n$. If, on the other hand, $\beta_n = 1$, then by assumption 2 of this theorem we have $S^n\beta \succeq \beta$, and therefore $f^n(p) = \pi_r(S^n\beta) \geq \pi_r(\beta) = p$. Again by the definition of the itinerary of p, we have $\omega_n = 1$ and hence $\omega_n = \beta_n$. □

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